MATHEMATICS

ON THE RIESZ SET OF A LINEAR OPERATOR

BY

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(Communicated by Prof. A. C. ZAANEN at the meeting of September 30, 1967)

1. NOTATION AND TERMINOLOGY; INTRODUCTION.

Throughout the present paper T will be a closed linear operator with domain $D(T)$ and range $R(T)$ in the complex Banach space X. The null space of T is denoted by $N(T)$.

The *dimension* of any linear space L is the maximal number of linearly independent elements in L. It will be denoted by dim L. Hence the value of dim L can be any non-negative integer or $+\infty$.

The dimension of the null space of T is called the *nullity* of T. It is denoted by $n(T)$. The dimension of the quotient space $X/R(T)$ is called the *defect* of T. It is denoted by $d(T)$. The *resolvent set* of T is the set

$$\varrho(T)=\{\lambda \in \mathbf{C}: n(\lambda-T)=d(\lambda-T)=0\},$$

where $\mathbf{C}$ denotes the set of complex numbers. The complement $\sigma(T)$ of $\varrho(T)$ in $\mathbf{C}$ is called the *spectrum* of T. The *Fredholm set* $\mathscr{F}(T)$ will be the set of all complex numbers λ such that $\lambda-T$ has finite nullity and defect. This terminology stems from the fact that a closed linear operator with finite defect has a closed range (cf. [4], IV. 1.13), and hence, for any λ in $\mathscr{F}(T)$, $\lambda-T$ is a so-called Fredholm operator.

For the definitions of the *ascent* $\alpha(T)$ and the *descent* $\delta(T)$ of T we refer to section 1 of A. E. TAYLOR's paper [11]. The *Riesz set* of T is the set

$$\mathscr{R}(T)=\{\lambda \in \mathbf{C}: \alpha(\lambda-T)<\infty,\ \delta(\lambda-T)<\infty\}.$$

In the present paper we investigate the spectral properties of the Riesz set $\mathscr{R}(T)$. One of our main tools is a stability theorem for operators of the form $(\lambda-T)^k$. Here k is a fixed natural number and λ is a complex variable. This stability theorem is formulated in section 2; it is a special case of a more general theorem due to K–H. FÖRSTER [2]. In section 3 we use this stability theorem in the study of the set $\mathscr{R}(T) \cap \mathscr{F}(T)$. In the case that T is densely defined it is shown that the points of $\mathscr{R}(T) \cap \mathscr{F}(T)$ either belong to the resolvent set of T or are poles of finite rank of the resolvent $(\lambda-T)^{-1}$. This is a considerable extension of a wrongly proved, but correct statement of S. R. CARADUS in [1]. If the resolvent set of T is not empty certain results in D. C. LAY's doctoral dissertation [8] imply that $\mathscr{R}(T) \cap \sigma(T)$ is a set of poles of the resolvent $(\lambda-T)^{-1}$. In the present paper we show that the same is true if the condition $\varrho(T) \neq \emptyset$ is replaced by the conditions $D(T)$ is dense in X and $\mathscr{F}(T)$ is not empty. The results in this paper are related to some recent work of M. SCHECHTER [10].

2. A STABILITY THEOREM.

Let T be a closed linear operator with domain and range in the Banach space X. Suppose that $n(T)$ and $d(T)$ are both finite. Then it follows (cf. [11], Lemma 3.3.) that T^i is a linear operator with

$$n(T^i) < \infty, \; d(T^i) < \infty \qquad (i = 0, 1, 2, \ldots).$$

Also, we can apply Theorem IV, 2.7 of [4] to show that the linear operator T^i is closed.

Let k be a positive integer, and take $0 \leqslant i \leqslant k$. Since T^i is a closed linear operator with domain $D(T^i) \supset D(T^k)$ it follows from Theorem V. 3.3. of [4] that T^i is a T^k-bounded linear operator, i.e., there exists a number M_i such that for all x in $D(T^k)$

$$\|T^i x\| \leqslant M_i(\|x\| + \|T^k x\|).$$

For any x in $D(T^k) = D((\lambda - T)^k)$, we have

$$\begin{aligned}(\lambda - T)^k x &= \textstyle\sum_{i=0}^{k} \lambda^i \{(-1)^{k-i} \binom{k}{i} T^{k-i}\} x \\ &= (-1)^k T^k x + \lambda A_1 x + \ldots + \lambda^k A_k x,\end{aligned}$$

where

$$A_i = (-1)^{k-i} \binom{k}{i} T^{k-i} \qquad (i = 1, \ldots, k).$$

Note that $D(T^k) \subset D(A_i)$ and, for x in $D(T^k)$,

$$\|A_i x\| \leqslant \binom{k}{i} M_{k-i} (\|x\| + \|T^k x\|).$$

Also, $R(T^k)$ is closed because T^k is a closed linear operator with finite defect. Combining these remarks it is readily seen that the linear operator

$$T(\lambda) = (\lambda - T)^k = (-1)^k T^k + \lambda A_1 + \ldots + \lambda^k A_k$$

with domain $D((\lambda - T)^k) = D(T^k)$ satisfies the conditions of Satz 2 of [2] (see also the last paragraph of [2]). Hence there exist a positive constant ϱ_k and a non-negative integer r_k such that for $0 < |\lambda| < \varrho_k$ the linear operator $(\lambda - T)^k$ is a closed linear operator with

(i) $n((\lambda - T)^k) = n(T^k) - r_k$,

(ii) $d((\lambda - T)^k) = d(T^k) - r_k$.

We summarize these facts in the following stability theorem.

THEOREM 1. *Let T be a closed linear operator with finite nullity and defect. Then there exist a positive constant ϱ_k and a non-negative integer r_k*

such that for $0<|\lambda|<\varrho_k$ *the linear operator* $(\lambda-T)^k$ *is a closed linear operator with*

(i) $n((\lambda-T)^k)=n(T^k)-r_k$,

(ii) $d((\lambda-T)^k)=d(T^k)-r_k$.

Remarks 1. Weaker versions of Theorem 1 have been proved by S. Kaniel and M. Schlechter ([7], Lemma 4.7), and more recently by S. R. Caradus ([1], formula 9).

2. For $k=1$ the preceding theorem is a special case of Theorem 5.1 (i) and (ii) in [5]. From this and the results of section 2 in [5] it follows that

$$r_1=\dim\,[N(T)/\{N(T)\cap D\}],$$

where

$$D=\bigcap_{n=1}^{\infty} R(T^n).$$

This implies quite easily that

$$r_1=\sup_n \dim\,[N(T)/\{N(T)\cap R(T^n)\}]. \tag{1}$$

If $k>1$ there is a similar, but more complicated formula for r_k. However, if T has a finite ascent everything gets very simple as is shown in the following lemma.

Lemma 2. *Let* T *be a closed linear operator with finite nullity and defect. If, in addition,* $\alpha(T)<\infty$ *then*

$$r_k=n(T^k) \qquad (k=1,2,\ldots).$$

Proof. Since (cf. [6], Lemma 3.1)

$$N(T)\cap R(T^i)\simeq\{N(T^{i+1})/N(T^i)\} \qquad (i=0,1,2,\ldots),$$

it follows that for $n\geqslant\alpha(T)$

$$N(T)\cap R(T^n)=(0).$$

Hence formula (1) implies that $r_1=\dim N(T)=n(T)$. But if $r_1=n(T)$, then $n(\lambda-T)=0$ for $0<|\lambda|<\varrho_1$, and hence

$$n((\lambda-T)^k)=0 \qquad (0<|\lambda|<\varrho_1,\ k=1,2,\ldots).$$

But then, by Theorem 1, $r_k=n(T^k)$ for $k=1,2,\ldots$.

3. The Riesz set.

Also in this section T will be a closed linear operator with domain and range in the Banach space X. The following theorem is an application of the stability theorem of section 2.

THEOREM 3. *Let $\lambda_0 \in \mathscr{R}(T) \cap \mathscr{F}(T)$, then there exists $\varepsilon > o$ such that for* $0 < |\lambda - \lambda_0| < \varepsilon$

(i) $n(\lambda - T) = 0$,

(ii) $\alpha(\lambda - T) = 0$,

(iii) $d(\lambda - T) = d(\lambda_0 - T) - n(\lambda_0 - T) = \dim\,[X/\{D((\lambda_0 - T)^q) + R(\lambda_0 - T)\}]$,

where $q = \delta(\lambda_0 - T)$,

(iv) $\delta(\lambda - T) \leqslant \delta(\lambda_0 - T)$; $\delta(\lambda - T)$ *is constant on the set* $0 < |\lambda - \lambda_0| < \varepsilon$.

In particular $\mathscr{R}(T) \cap \mathscr{F}(T)$ *is open.*

Proof. Without loss of generality we may suppose that $\lambda_0 = 0$. So T is a closed linear operator with finite nullity and defect. Hence, by Theorem 1, there exist $\varrho_k > 0$ and a non-negative integer r_k such that for $0 < |\lambda| < \varrho_k$

(i) $n((\lambda - T)^k) = n(T^k) - r_k$,

(ii) $d((\lambda - T)^k) = d(T^k) - r_k$.

Since $\alpha(T)$ is finite, Lemma 2 implies that $r_k = n(T^k)$ for $k = 1, 2, \ldots$, and thus

$$n(\lambda - T) = 0, \ \alpha(\lambda - T) = 0 \qquad (0 < |\lambda| < \varrho_1).$$

Also, for $k = 1, 2, \ldots$ and $0 < |\lambda| < \varrho_k$,

$$d((\lambda - T)^k) = d(T^k) - n(T^k). \tag{2}$$

Applying [6], Theorem 4.5 it is readily seen that

$$d(T) - n(T) \leqslant \dim\,[X/\{D(T^q) + R(T)\}], \tag{3}$$

where $q = \delta(T)$, and we have equality in formula (3) if T has the additional property $N(T) \cap R(T^q) = (0)$. Since $q = \delta(T) \geqslant \alpha(T)$ (cf. [6], Theorem 4.1.) Lemma 3.4 (b) in [11] implies that this is the case. So

$$d(\lambda - T) = d(T) - n(T) = \dim\,[X/\{D(T^q) + R(T)\}]$$

for $0 < |\lambda| < \varrho_1$.

Let $\varepsilon = \min\,\{\varrho_1, \ldots, \varrho_q, \varrho_{q+1}\}$ and take $0 < |\lambda| < \varepsilon$. Then (see formula (2))

$$d((\lambda - T)^k) = d(T^k) - n(T^k) \tag{4}$$

for $k = 1, \ldots, q, q+1$. Since $q = \delta(T) \geqslant \alpha(T)$ the preceding formula implies

$$d((\lambda - T)^{q+1}) = d((\lambda - T)^q)$$

and thus $\delta(\lambda - T) \leqslant q = \delta(T)$. Also, formula (4) shows that $\delta(\lambda - T)$ is the least non-negative integer r with

$$d(T^r) - n(T^r) = d(T^{r+1}) - n(T^{r+1}).$$

Hence $\delta(\lambda - T)$ is constant on $0 < |\lambda| < \varepsilon$. This completes the proof.

Remarks 3. In a slightly more general setting Theorem 3 (ii) (and thus (i) too) has been proved by A. E. TAYLOR ([11], Theorem 9.4; see also [8], Lemma 2.10). However, our methods differ from those used in [11]. The fact that $\mathscr{R}(T) \cap \mathscr{F}(T)$ is an open set is already contained in [1].

4. If λ_0 in Theorem 3 satisfies the additional condition $\alpha(\lambda_0 - T) < \delta(\lambda_0 - T)$ then it follows easily from the arguments used in the last part of the preceding proof that $\delta(\lambda - T) = \delta(\lambda_0 - T)$ in an open neighbourhood of λ_0. If $\alpha(\lambda_0 - T) = \delta(\lambda_0 - T)$ we may have equality as well as inequality in Theorem 3 (iv). The following examples illustrate this point.

(A) Let $X = \mathbf{C}^3$, and

$$D(T) = \{(x_1, x_2, x_3) \in X : x_3 = 0\}.$$

Define T on $D(T)$ by $T(x_1, x_2, 0) = (x_1, 0, 0)$. Then

$$n(T) = 1,\ d(T) = 2, \qquad \alpha(T) = \delta(T) = 1,$$

and for $0 < |\lambda| < 1$ we have

$$n(\lambda - T) = 0,\ d(\lambda - T) = 1,\ \alpha(\lambda - T) = 0,\ \delta(\lambda - T) = 1.$$

(B) Let $X = \mathbf{C}^2$, and define T on X by $T(x_1, x_2) = (x_1, 0)$. Then

$$n(T) = d(T) = 1, \qquad \alpha(T) = \delta(T) = 1,$$

and for $0 < |\lambda| < 1$ we have

$$n(\lambda - T) = d(\lambda - T) = 0, \qquad \alpha(\lambda - T) = \delta(\lambda - T) = 0.$$

We proceed with the special case $\overline{D(T)} = X$. CARADUS ([1], Lemma 2) showed that in the case that T is densely defined

$$n(\lambda - T) = d(\lambda - T) < \infty, \qquad \alpha(\lambda - T) = \delta(\lambda - T) < \infty \tag{5}$$

for any λ in $\mathscr{R}(T) \cap \mathscr{F}(T)$. Using this fact Theorem 3 implies that, given λ_0 in $\mathscr{R}(T) \cap \mathscr{F}(T)$, there exists $\varepsilon > 0$ such that $d(\lambda - T) = n(\lambda - T) = 0$ for $0 < |\lambda - \lambda_0| < \varepsilon$, i.e.,

$$\{\lambda : 0 < |\lambda - \lambda_0| < \varepsilon\} \subset \varrho(T).$$

Hence either $\lambda_0 \in \varrho(T)$ or λ_0 is an isolated point in $\sigma(T)$ with the additional properties (5). Applying Theorem 9.6 in [11] it follows that those points are poles of finite rank of the resolvent operator $(\lambda - T)^{-1}$. So we have the following corollary.

Corollary 4. *If T is densely defined, then*

$$\mathscr{R}(T) \cap \mathscr{F}(T) \cap \sigma(T)$$

is a set of isolated points in $\mathscr{R}(T) \cap \mathscr{F}(T)$ each of which is a pole of finite rank of the resolvent $(\lambda - T)^{-1}$.

Remarks 5. A much weaker version of the preceding corollary has been stated by CARADUS [1] (Theorem 2 and Corollary in [1]). However CARADUS' result is incorrectly proven, because his proof depends on a lemma (Lemma 3 in [1]) which is not true. To show this we present the following example.

Let T be the linear operator on $\mathbf{C}^2$ defined by

$$T(x_1, x_2) = (x_1, o).$$

Further, define on $\mathbf{C}^2$ the linear operators S_1 and S_2 by

$$S_1(x_1, x_2) = (x_1, x_2), \qquad S_2(x_1, x_2) = (x_1 + x_2, o).$$

Then, for $0 < |\lambda| < 1$,

$$n(T + \lambda S_1) = 0, \qquad n(T + \lambda S_2) = 1.$$

Hence, there does not exist $\varepsilon > 0$ such that $n(T+S)$ is constant for $0 < \|S\| < \varepsilon$. Clearly, this contradicts Lemma 3 in [1].

6. In his work on the essential spectrum of a linear operator D. C. LAY proved a theorem which contains Corollary 4 (cf. [9], Theorem 1 (2) and (3)). LAY's theorem and our corollary were obtained independently.

It is interesting to observe that much more can be said about the Riesz set than is done in Corollary 4. In [8] D. C. LAY showed that if T is a closed (not necessarily densely defined) linear operator with $\varrho(T)$ not empty, then $\lambda_0 \in \sigma(T)$, $\alpha(\lambda_0 - T) < \infty$ and $\delta(\lambda_0 - T) < \infty$ imply that λ_0 is an isolated point of $\sigma(T)$ and is a pole of the resolvent $(\lambda - T)^{-1}$ ([8], Theorem 2.2). Hence, if $\varrho(T) \neq \emptyset$, then $\mathscr{R}(T)$ is an open set and $\mathscr{R}(T) \cap \sigma(T)$ is a set of poles of the resolvent operator of T. In the present paper we shall show that the same conclusion holds if the condition $\varrho(T) \neq \emptyset$ is replaced by the conditions $\mathscr{F}(T)$ not empty and $D(T)$ dense in X. The main reason for this is the following lemma.

Lemma 5. *If T is a densely defined closed linear operator and if $\mathscr{F}(T)$ is not empty, then*

$$D(T^i) + R(T^k) = X \qquad (i, k = 0, 1, 2, \ldots).$$

Proof. Take $z \in \mathscr{F}(T)$. Then $z - T$ is a closed densely defined linear operator with finite nullity and defect. Hence (cf. [4], Theorem IV. 2.7) the same is true for $(z-T)^i$ $(i = 0, 1, 2, \ldots)$. Further, $d((z-T)^i) < \infty$ implies that $R((z-T)^i)$ is a closed subspace of X (cf. [4], Corollary IV. 1.13).

Let i and j be non-negative integers. Since $D((z-T)^i)$ is dense in X, and since $R((z-T)^j)$ is a closed subspace of finite codimension in X it is easy to show (cf. [3], Lemma 2.1.) that

$$D((z-T)^i) + R((z-T)^j) = X. \tag{6}$$

Note that $D((z-T)^i)=D(T^i)$. Further, we have

$$R((z-T)^{i+k}) \subset D(T^i)+R(T^k)$$

for any non-negative integer k. Combining the last two facts with formula (6) we obtain the desired result.

Corollary 6. *If T is a densely defined closed linear operator and if $\mathscr{F}(T)$ is not empty, then $\alpha(T)<\infty$, $\delta(T)<\infty$ imply $p=\alpha(T)=\delta(T)$ and*

$$X=R(T^p) \oplus N(T^p).$$

Proof. The proof of this statement is exactly the same as the proof of Theorem 1.6 in [8].

THEOREM 7. *If T is a densely defined closed linear operator, and if $\mathscr{F}(T)$ is not empty, then $\mathscr{R}(T)$ is an open set and*

$$\mathscr{R}(T) \cap \sigma(T)$$

is a set of poles of the resolvent $(\lambda-T)^{-1}$.

Proof. Take λ_0 in $\mathscr{R}(T)$. If $\lambda_0 \in \varrho(T)$, there exists an open neighbourhood of λ_0 which is contained in $\varrho(T)$. Since $\varrho(T) \subset \mathscr{R}(T)$ it follows that λ_0 is an interior point of $\mathscr{R}(T)$. If $\lambda_0 \in \sigma(T)$ it is more difficult to show that λ_0 is an interior point of $\mathscr{R}(T)$.

From the preceding corollary we know that

$$p=\alpha(\lambda_0-T)=\delta(\lambda_0-T)<\infty \tag{7}$$

and

$$R((\lambda_0-T)^p) \oplus N((\lambda_0-T)^p)=X. \tag{8}$$

First of all we show that both subspaces in the decomposition (8) are closed. Since $\mathscr{F}(T)$ is not empty, any polynomial in T is a closed linear operator (cf. [4], Corollary IV. 2.12). In particular $(\lambda_0-T)^p$ is a closed linear operator. But then, $N((\lambda_0-T)^p)$ being the null space of a closed linear operator will be closed in X. Further this shows that $R((\lambda_0-T)^p)$ has a closed complementary subspace in X, and so, by [4] Theorem IV. 1.12, $R((\lambda_0-T)^p)$ is a closed subspace of X.

Secondly, we show that $\lambda_0 \in \sigma(T)$ implies that λ_0 is an isolated point of $\sigma(T)$. Let $X_1=R((\lambda_0-T)^p)$ and $X_2=N((\lambda_0-T)^p)$. Then X_1 and X_2 are Banach spaces, and T is completely reduced by these subspaces. Let T_i denote the restriction of T to X_i $(i=1, 2)$. The linear operator λ_0-T_1, acting on X_1, is closed and also, by formula (7), one-one and onto. So $\lambda_0 \in \varrho(T_1)$. Since the resolvent set of any closed linear operator acting on a Banach space is open, these exists $\varepsilon>o$ such that $\{\lambda\colon |\lambda-\lambda_0|<\varepsilon\} \subset \varrho(T_1)$. The linear operator λ_0-T_2, acting on X_2, is nilpotent. So $\{\lambda\colon o<|\lambda-\lambda_0|\} \subset$

$\subset \varrho(T_2)$. Since $\lambda_0 - T$ is completely reduced by X_1 and X_2, this implies

$$\{\lambda\colon 0 < |\lambda - \lambda_0| < \varepsilon\} \subset \varrho(T_1) \cap \varrho(T_2) = \varrho(T).$$

Hence λ_0 is an isolated point of $\sigma(T)$. Also, it shows that λ_0 is an interior point of $\mathscr{R}(T)$. Thus any point of $\mathscr{R}(T)$ is interior point of $\mathscr{R}(T)$, i.e., $\mathscr{R}(T)$ is open.

If $\lambda_0 \in \mathscr{R}(T) \cap \sigma(T)$, then λ_0 is an isolated point of $\sigma(T)$, $p = \alpha(\lambda_0 - T) = \delta(\lambda_0 - T) < \infty$ and $R((\lambda_0 - T)^p)$ is closed. Hence, by Theorem 9.1 in [11], λ_0 is a pole of $(\lambda - T)^{-1}$.

We conclude with two final remarks. In a letter to the author discussing the results of this paper Dr. Lay wrote that he was aware of the fact that the condition $\varrho(T) \neq \emptyset$ in [8], Theorem 2.2 could be replaced by the conditions $D(T)$ is dense in X and $\mathscr{F}(T)$ is not empty without invalidating the theorem.

In section 4 of [10] M. Schechter introduces the following numbers

$$r(T) = \lim_{k \to \infty} n(T^k), \qquad r^*(T) = \lim_{k \to \infty} d(T^k),$$

and in the same section he studies Fredholm operators T with $r(T)$ and $r^*(T)$ finite. Clearly, $r(T) < \infty$ implies $\alpha(T) < \infty$ and $r^*(T) < \infty$ implies $\delta(T) < \infty$. Keeping this in mind it is easily seen that some results of Schlechter's work are extended in this paper.

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